Stability of Proximal Average Relaxation for Signal Recovery

Image and Audio Reconstruction under Noise and Operator Perturbations

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What Is Signal Recovery?

- An image, audio waveform, or any other vector is represented as a signal $\bar{x} \in \mathbb{R}^N$.
- The reconstruction is restricted to an **admissible signal set** $C \subseteq \mathbb{R}^N$. (e.g., $C = [-1, 1]$ or $C = [0, 255]$ and other such hypercubes)

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- Two-staged signal transformation model:

$$\boxed{\text{True Signal } \bar{x}} \xrightarrow{\bar{L}_i} \boxed{\text{Linear Stage}} \xrightarrow{\bar{F}_i} \boxed{\text{Nonlinear Stage}} \longrightarrow \boxed{\text{Observed Data } \bar{p}_i}$$

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- **Linear Operators ($\bar{L}_i$):** Blur, convolution, or audio echo.
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Inverse Problem: Use the observed data $\bar{p}_i$ to recover $\bar{x}$.

Why Relaxation Is Needed

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  - Observed data p_i is perturbed by noise.
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Ideal Exact Model → Available Perturbed Model

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Ideal Exact Model		Available Perturbed Model
$\boxed{\bar{F}_i(\bar{L}_i\bar{x}) = \bar{p}_i}$	$\longrightarrow$	$\boxed{F_i(L_i x) = p_i}$

- **Inconsistency:** No exact signal $\bar{x}$ satisfies every perturbed observation equation.
- Meaningful, relaxed recovery methods already exist.
- This research investigates the additional quantitative sensitivity estimate relating perturbations to recovery error.

The Proximal Average Relaxation

- For each perturbed observation, define a **correction operator** that adjusts x based on the mismatch in each observation.

$$T_i x = x - \frac{1}{\|L_i\|_{\mathrm{op}}^2} L_i^* (F_i(L_i x) - p_i)$$

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- Combine the correction operators using reliability weights: Reliable observations are assigned larger weights.

$$T = \sum_{i=1}^{m} \omega_i T_i, \quad \omega_i > 0, \quad \sum_{i=1}^{m} \omega_i = 1.$$

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- The **proximal average relaxation** produces a relaxed signal x^* that is a fixed point of the averaged operator T:

$$\boxed{x^* \in \operatorname{Fix} T \quad \Longleftrightarrow \quad T x^* = x^*.}$$

Perturbations and Research Question

- To analyze the reconstruction stability of x^*, we measure three types of perturbations in the recovery model:

Operator Mismatch (γ_i): $\|\bar{L}_i y - L_i y\| \leq \gamma_i \|y\|, \quad (y \in \mathcal{H})$

Nonlinear Mismatch (λ_i): $\|\bar{F}_i(u) - F_i(u)\| \leq \lambda_i \|u\|, \quad (u \in \mathcal{G}_i)$

Observation Noise ($\|\delta_i\|$): $\delta_i = p_i - \bar{p}_i \implies \|p_i - \bar{p}_i\| = \|\delta_i\|$

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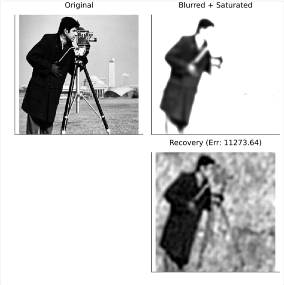
- Main Research Question:

$$\boxed{\text{Can we explicitly bound } \|\bar{x} - x^*\| \text{ using } \gamma_i, \lambda_i, \text{ and } \|\delta_i\|?}$$

Examples

These are two different examples of how different levels of noise and operator "guessing" affect the reconstructed images.



A Linear Sensitivity Bound

Assumptions:

- Each perturbed nonlinear operator F_i is firmly nonexpansive.
- The perturbations satisfy the previously stated bounds.
- The perturbed linear operators satisfy
$$\|L_i\|_{\mathrm{op}} \geq \ell_i > 0.$$
- The perturbed averaged operator T is globally contractive on $\mathcal{H}$:
$$\|Tx - Ty\| \leq q\|x - y\|, \qquad q \in [0, 1).$$
Following from global contractivity, T has a unique fixed point x^*:
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Then we obtain the sensitivity bound:

$$\boxed{\|\bar{x} - x^*\| \leq \frac{1}{1-q} \sum_{i=1}^{m} \omega_i \frac{1}{\ell_i} (\|\bar{x}\|\gamma_i + \lambda_i\|\bar{x}\|\|\bar{L}_i\|_{\text{op}} + \|\delta_i\|).}$$

Proof Strategy

- Express the exact and recovered signals as fixed points of their respective operators: $\bar{T}\bar{x} = \bar{x}$ and $Tx^* = x^*$.
- Using this fixed point framework and applying contractivity,

$$\begin{aligned}\|\bar{x} - x^*\| &= \|\bar{T}\bar{x} - Tx^*\| \\ &\leq \|\bar{T}\bar{x} - T\bar{x}\| + q\|\bar{x} - x^*\|.\end{aligned}$$

  Rearranging gives,

$$\|\bar{x} - x^*\| \leq \frac{1}{1-q}\|\bar{T}\bar{x} - T\bar{x}\|.$$

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Rearranging gives,
$$\|\bar{x} - x^*\| \leq \frac{1}{1-q}\|\bar{T}\bar{x} - T\bar{x}\|.$$
- Bound the averaged operator estimate:
$$\|\bar{T}\bar{x} - T\bar{x}\| \leq \sum_{i=1}^{m} \omega_i \|\bar{x} - T_i\bar{x}\|.$$
- Applying the perturbation bounds gives the component bounds:
$$\|\bar{x} - T_i\bar{x}\| \leq \frac{1}{\ell_i}(\gamma_i\|\bar{x}\| + \lambda_i\|\bar{x}\|\|\bar{L}_i\|_{\text{op}} + \|\delta_i\|).$$
- Substituting the component bounds into the averaged operator estimate gives the final sensitivity theorem.

Tikhonov Regularization Supplies Contractivity

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- However, while some operators we researched are strong monotone, many operators arising in signal recovery are only *firmly non-expansive* (monotone), which guarantees stability but not a contraction. Thus we use Tikhonov regularization which adds a penalty term to the recovery problem (ϵId). This guarantees contractivity.

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Image Reconstruction Test

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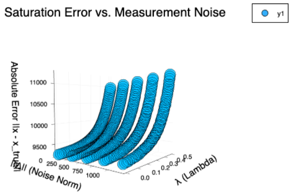
Our main goal for this code is to test the question: Does the reconstruction error grow *linearly* as predicted by our theory?

Numerical Results

Without Regularization



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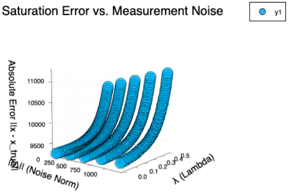

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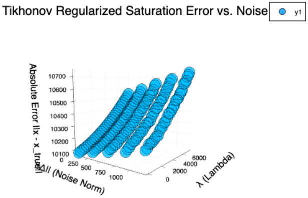
With Regularization



Regularization enforces contractivity, confirming that recovery error scales linearly with mismatch and noise.

Conclusion & Future Work

What We Achieved

- Developed an explicit quantitative sensitivity bound for proximal average relaxation.
- Showed reconstruction error grows linearly with measurement/model perturbations.
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Next Steps

- Extend the sensitivity analysis to incorporate Tikhonov regularization directly, replacing the contractivity assumption with firm non-expansiveness.
- Develop sensitivity bounds that do not rely on contractivity, broadening the class of admissible operators.
- Validate the theoretical results through additional numerical experiments.

Citations

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